Challenge Problems

Chapter 3

A Click here for answers. S Click here for solutions.

1. (a) Find the domain of the function $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$.
(b) Find $f'(x)$.
(c) Check your work in parts (a) and (b) by graphing f and f' on the same screen.

Chapter 4

A Click here for answers. S Click here for solutions.

1. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

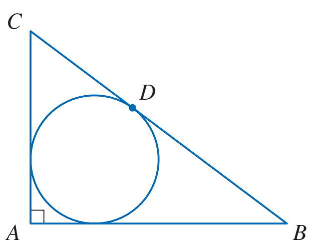


FIGURE FOR PROBLEM 2

2. (a) Let ABC be a triangle with right angle A and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at D, show that

$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

(b) If $\theta = \frac{1}{2}\angle C$, express the radius r of the inscribed circle in terms of a and θ.
(c) If a is fixed and θ varies, find the maximum value of r.

3. A triangle with sides a, b, and c varies with time t, but its area never changes. Let θ be the angle opposite the side of length a and suppose θ always remains acute.
(a) Express $d\theta/dt$ in terms of b, c, θ, db/dt, and dc/dt.
(b) Express da/dt in terms of the quantities in part (a).

Chapter 5

A Click here for answers. S Click here for solutions.

1. In Sections 5.1 and 5.2 we used the formulas for the sums of the kth powers of the first n integers when $k = 1, 2$, and 3. (These formulas are proved in Appendix E.) In this problem we derive formulas for any k. These formulas were first published in 1713 by the Swiss mathematician James Bernoulli in his book *Ars Conjectandi*.
(a) The **Bernoulli polynomials** B_n are defined by $B_0(x) = 1$, $B_n'(x) = B_{n-1}(x)$, and $\int_0^1 B_n(x)\,dx = 0$ for $n = 1, 2, 3, \ldots$. Find $B_n(x)$ for $n = 1, 2, 3$, and 4.
(b) Use the Fundamental Theorem of Calculus to show that $B_n(0) = B_n(1)$ for $n \geqslant 2$.
(c) If we introduce the **Bernoulli numbers** $b_n = n!\,B_n(0)$, then we can write

$$B_0(x) = b_0 \qquad\qquad B_1(x) = \frac{x}{1!} + \frac{b_1}{1!}$$

$$B_2(x) = \frac{x^2}{2!} + \frac{b_1}{1!}\frac{x}{1!} + \frac{b_2}{2!} \qquad\qquad B_3(x) = \frac{x^3}{3!} + \frac{b_1}{1!}\frac{x^2}{2!} + \frac{b_2}{2!}\frac{x}{1!} + \frac{b_3}{3!}$$

and, in general,

$$B_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} b_k x^{n-k} \qquad \text{where} \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

[The numbers $\binom{n}{k}$ are the binomial coefficients.] Use part (b) to show that, for $n \geqslant 2$,

$$b_n = \sum_{k=0}^{n}\binom{n}{k} b_k$$

and therefore

$$b_{n-1} = -\frac{1}{n}\left[\binom{n}{0}b_0 + \binom{n}{1}b_1 + \binom{n}{2}b_2 + \cdots + \binom{n}{n-2}b_{n-2}\right]$$

This gives an efficient way of computing the Bernoulli numbers and therefore the Bernoulli polynomials.

(d) Show that $B_n(1-x) = (-1)^n B_n(x)$ and deduce that $b_{2n+1} = 0$ for $n > 0$.

(e) Use parts (c) and (d) to calculate b_6 and b_8. Then calculate the polynomials B_5, B_6, B_7, B_8, and B_9.

(f) Graph the Bernoulli polynomials $B_1, B_2, \ldots, B_9$ for $0 \leqslant x \leqslant 1$. What pattern do you notice in the graphs?

(g) Use mathematical induction to prove that $B_{k+1}(x+1) - B_{k+1}(x) = x^k/k!$.

(h) By putting $x = 0, 1, 2, \ldots, n$ in part (g), prove that

$$1^k + 2^k + 3^k + \cdots + n^k = k!\,[B_{k+1}(n+1) - B_{k+1}(0)] = k! \int_0^{n+1} B_k(x)\,dx$$

(i) Use part (h) with $k = 3$ and the formula for B_4 in part (a) to confirm the formula for the sum of the first n cubes in Section 5.2.

(j) Show that the formula in part (h) can be written symbolically as

$$1^k + 2^k + 3^k + \cdots + n^k = \frac{1}{k+1}[(n+1+b)^{k+1} - b^{k+1}]$$

where the expression $(n+1+b)^{k+1}$ is to be expanded formally using the Binomial Theorem and each power b^i is to be replaced by the Bernoulli number b_i.

(k) Use part (j) to find a formula for $1^5 + 2^5 + 3^5 + \cdots + n^5$.

Chapter 6

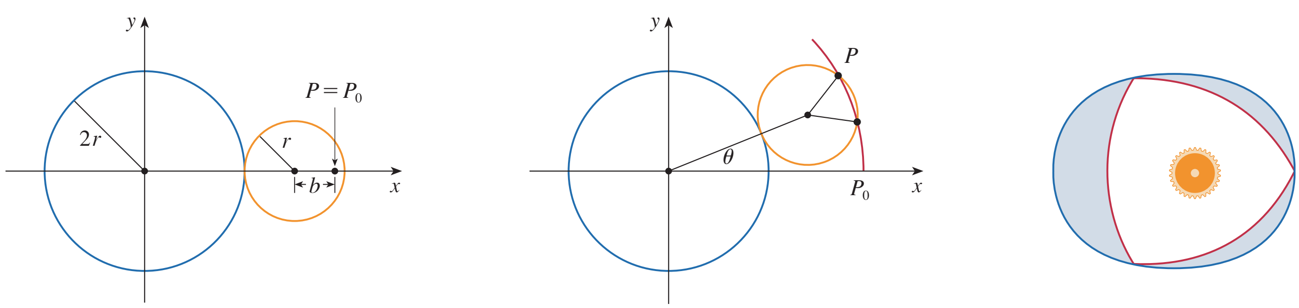
A Click here for answers. S Click here for solutions.

1. A solid is generated by rotating about the x-axis the region under the curve $y = f(x)$, where f is a positive function and $x \geqslant 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for all $b > 0$. Find the function f.

Chapter 10

A Click here for answers. S Click here for solutions.

1. A circle C of radius $2r$ has its center at the origin. A circle of radius r rolls without slipping in the counterclockwise direction around C. A point P is located on a fixed radius of the rolling circle at a distance b from its center, $0 < b < r$. [See parts (i) and (ii) of the figure.] Let L be the line from the center of C to the center of the rolling circle and let θ be the angle that L makes with the positive x-axis.



(i) (ii) (iii)

(a) Using θ as a parameter, show that parametric equations of the path traced out by P are

$$x = b\cos 3\theta + 3r\cos\theta \qquad y = b\sin 3\theta + 3r\sin\theta$$

Note: If $b = 0$, the path is a circle of radius $3r$; if $b = r$, the path is an *epicycloid.* The path traced out by P for $0 < b < r$ is called an *epitrochoid.*

(b) Graph the curve for various values of b between 0 and r.

(c) Show that an equilateral triangle can be inscribed in the epitrochoid and that its centroid is on the circle of radius b centered at the origin.

Note: This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the curve.

(d) In most rotary engines the sides of the equilateral triangles are replaced by arcs of circles centered at the opposite vertices as in part (iii) of the figure. (Then the diameter of the rotor is constant.) Show that the rotor will fit in the epitrochoid if $b \leqslant 3(2 - \sqrt{3})r/2$.

Chapter 11

S Click here for solutions.

1. (a) Show that, for $n = 1, 2, 3, \ldots,$

$$\sin \theta = 2^n \sin \frac{\theta}{2^n} \cos \frac{\theta}{2} \cos \frac{\theta}{4} \cos \frac{\theta}{8} \cdots \cos \frac{\theta}{2^n}$$

(b) Deduce that

$$\frac{\sin \theta}{\theta} = \cos \frac{\theta}{2} \cos \frac{\theta}{4} \cos \frac{\theta}{8} \cdots$$

The meaning of this infinite product is that we take the product of the first n factors and then we take the limit of these partial products as $n \to \infty$.

(c) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots$$

This infinite product is due to the French mathematician François Viète (1540–1603). Notice that it expresses π in terms of just the number 2 and repeated square roots.

2. Suppose that $a_1 = \cos \theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, $b_1 = 1$, and

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n) \qquad b_{n+1} = \sqrt{b_n a_{n+1}}$$

Use Problem 1 to show that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \frac{\sin \theta}{\theta}$$

Answers

Chapter 3

S Solutions

1. (a) $[-1, 2]$ (b) $-1/\left(8\sqrt{3-x}\sqrt{2-\sqrt{3-x}}\sqrt{1-\sqrt{2-\sqrt{3-x}}}\right)$

Chapter 4

S Solutions

1. $\frac{4}{3}$

3. (a) $-\tan\theta\left[\frac{1}{c}\frac{dc}{dt}+\frac{1}{b}\frac{db}{dt}\right]$ (b) $\dfrac{b\dfrac{db}{dt}+c\dfrac{dc}{dt}-\left(b\dfrac{dc}{dt}+c\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2+c^2-2bc\cos\theta}}$

Chapter 5

S Solutions

1. (a) $B_1(x)=x-\frac{1}{2}$, $B_2(x)=\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{12}$, $B_3(x)=\frac{1}{6}x^3-\frac{1}{4}x^2+\frac{1}{12}x$, $B_4(x)=\frac{1}{24}x^4-\frac{1}{12}x^3+\frac{1}{24}x^2-\frac{1}{720}$

(e) $b_6=\frac{1}{42}$, $b_8=-\frac{1}{30}$;

$B_5(x)=\frac{1}{120}\left(x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x\right)$,

$B_6(x)=\frac{1}{720}\left(x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}\right)$,

$B_7(x)=\frac{1}{5040}\left(x^7-\frac{7}{2}x^6+\frac{7}{2}x^5-\frac{7}{6}x^3+\frac{1}{6}x\right)$,

$B_8(x)=\frac{1}{40,320}\left(x^8-4x^7+\frac{14}{3}x^6-\frac{7}{3}x^4+\frac{2}{3}x^2-\frac{1}{30}\right)$,

$B_9(x)=\frac{1}{362,880}\left(x^9-\frac{9}{2}x^8+6x^7-\frac{21}{5}x^5+2x^3-\frac{3}{10}x\right)$

(f) There are four basic shapes for the graphs of B_n (excluding B_1), and as n increases, they repreat in a cycle of four. For $n=4m$, the shape resembles that of the graph of $-\cos 2\pi x$; for $n=4m+1$, that of $-\sin 2\pi x$; for $n=4m+2$, that of $\cos 2\pi x$; and for $n=4m+3$, that of $\sin 2\pi x$.

(k) $\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

Chapter 6

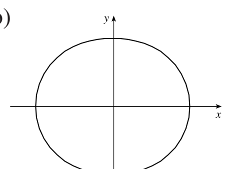
S Solutions

1. $f(x)=\sqrt{2x/\pi}$

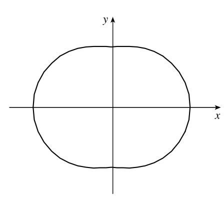
Chapter 10

S Solutions

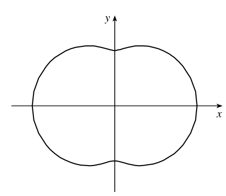
1. (b)



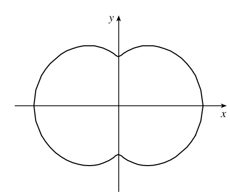
$b=\frac{1}{5}r$



$b=\frac{2}{5}r$



$b=\frac{3}{5}r$



$b=\frac{4}{5}r$

Solutions

E Exercises Chapter 3

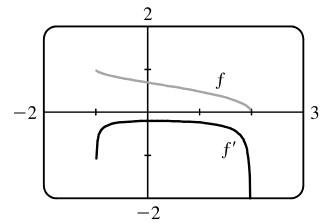
1. (a) $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$

$$D = \left\{x \mid 3 - x \geq 0, 2 - \sqrt{3 - x} \geq 0, 1 - \sqrt{2 - \sqrt{3 - x}} \geq 0\right\} = \left\{x \mid 3 \geq x, 2 \geq \sqrt{3 - x}, 1 \geq \sqrt{2 - \sqrt{3 - x}}\right\}$$
$$= \{x \mid 3 \geq x,\ 4 \geq 3 - x, 1 \geq 2 - \sqrt{3 - x}\} = \{x \mid x \leq 3, x \geq -1, 1 \leq \sqrt{3 - x}\}$$
$$= \{x \mid x \leq 3, x \geq -1, 1 \leq 3 - x\} = \{x \mid x \leq 3, x \geq -1, x \leq 2\} = \{x \mid -1 \leq x \leq 2\} = [-1, 2]$$

(b) $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}} \quad \Rightarrow$

$$f'(x) = \frac{1}{2\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}} \frac{d}{dx}\left(1 - \sqrt{2 - \sqrt{3 - x}}\right)$$
$$= \frac{1}{2\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}} \cdot \frac{-1}{2\sqrt{2 - \sqrt{3 - x}}} \frac{d}{dx}\left(2 - \sqrt{3 - x}\right)$$
$$= -\frac{1}{8\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}\,\sqrt{2 - \sqrt{3 - x}}\,\sqrt{3 - x}}$$

(c)



Note that f is always decreasing and f' is always negative.

E Exercises Chapter 4

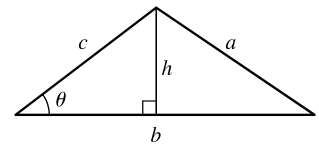
1. $f(x) = \dfrac{1}{1 + |x|} + \dfrac{1}{1 + |x - 2|}$

$$= \begin{cases} \dfrac{1}{1 - x} + \dfrac{1}{1 - (x - 2)} & \text{if } x < 0 \\ \dfrac{1}{1 + x} + \dfrac{1}{1 - (x - 2)} & \text{if } 0 \leq x < 2 \\ \dfrac{1}{1 + x} + \dfrac{1}{1 + (x - 2)} & \text{if } x \geq 2 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} \dfrac{1}{(1 - x)^2} + \dfrac{1}{(3 - x)^2} & \text{if } x < 0 \\ \dfrac{-1}{(1 + x)^2} + \dfrac{1}{(3 - x)^2} & \text{if } 0 < x < 2 \\ \dfrac{-1}{(1 + x)^2} - \dfrac{1}{(x - 1)^2} & \text{if } x > 2 \end{cases}$$

We see that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 2$. For $0 < x < 2$, we have

$f'(x) = \dfrac{1}{(3 - x)^2} - \dfrac{1}{(x + 1)^2} = \dfrac{(x^2 + 2x + 1) - (x^2 - 6x + 9)}{(3 - x)^2(x + 1)^2} = \dfrac{8(x - 1)}{(3 - x)^2(x + 1)^2}$, so $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$ and $f'(x) > 0$ for $1 < x < 2$. We have shown that $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$; and $f'(x) < 0$ for $x > 2$. Therefore, by the First Derivative Test, the local maxima of f are at $x = 0$ and $x = 2$, where f takes the value $\frac{4}{3}$. Therefore, $\frac{4}{3}$ is the absolute maximum value of f.

3. (a) $A = \frac{1}{2}bh$ with $\sin\theta = h/c$, so $A = \frac{1}{2}bc\sin\theta$. But A is a constant, so differentiating this equation with respect to t, we get



$$\frac{dA}{dt} = 0 = \frac{1}{2}\left[bc\cos\theta\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\sin\theta + \frac{db}{dt}\,c\sin\theta\right] \quad\Rightarrow$$

$$bc\cos\theta\,\frac{d\theta}{dt} = -\sin\theta\left[b\,\frac{dc}{dt} + c\,\frac{db}{dt}\right] \quad\Rightarrow\quad \frac{d\theta}{dt} = -\tan\theta\left[\frac{1}{c}\frac{dc}{dt} + \frac{1}{b}\frac{db}{dt}\right].$$

(b) We use the Law of Cosines to get the length of side a in terms of those of b and c, and then we differentiate implicitly with respect to t: $a^2 = b^2 + c^2 - 2bc\cos\theta \quad\Rightarrow\quad 2a\,\frac{da}{dt} = 2b\,\frac{db}{dt} + 2c\,\frac{dc}{dt} - 2\left[bc(-\sin\theta)\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\cos\theta + \frac{db}{dt}\,c\cos\theta\right]$

$\Rightarrow\quad \dfrac{da}{dt} = \dfrac{1}{a}\left(b\,\dfrac{db}{dt} + c\,\dfrac{dc}{dt} + bc\sin\theta\,\dfrac{d\theta}{dt} - b\,\dfrac{dc}{dt}\cos\theta - c\,\dfrac{db}{dt}\cos\theta\right)$. Now we substitute our value of a from the Law of Cosines and the value of $d\theta/dt$ from part (a), and simplify (primes signify differentiation by t):

$$\frac{da}{dt} = \frac{bb' + cc' + bc\sin\theta\,[-\tan\theta\,(c'/c + b'/b)] - (bc' + cb')(\cos\theta)}{\sqrt{b^2 + c^2 - 2bc\,\cos\theta}}$$

$$= \frac{bb' + cc' - [\sin^2\theta\,(bc' + cb') + \cos^2\theta\,(bc' + cb')]/\cos\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}} = \frac{bb' + cc' - (bc' + cb')\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}$$

E Exercises Chapter 5

1. (a) To find $B_1(x)$, we use the fact that $B_1'(x) = B_0(x) \quad\Rightarrow\quad B_1(x) = \int B_0(x)\,dx = \int 1\,dx = x + C$. Now we impose the condition that $\int_0^1 B_1(x)\,dx = 0 \quad\Rightarrow\quad 0 = \int_0^1 (x + C)\,dx = \left[\frac{1}{2}x^2\right]_0^1 + [Cx]_0^1 = \frac{1}{2} + C \quad\Rightarrow\quad C = -\frac{1}{2}$. So $B_1(x) = x - \frac{1}{2}$. Similarly $B_2(x) = \int B_1(x)\,dx = \int\left(x - \frac{1}{2}\right)dx = \frac{1}{2}x^2 - \frac{1}{2}x + D$. But $\int_0^1 B_2(x)\,dx = 0 \quad\Rightarrow\quad 0 = \int_0^1\left(\frac{1}{2}x^2 - \frac{1}{2}x + D\right)dx = \frac{1}{6} - \frac{1}{4} + D \quad\Rightarrow\quad D = \frac{1}{12}$, so $B_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$. $B_3(x) = \int B_2(x)\,dx = \int\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}\right)dx = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x + E$. But $\int_0^1 B_3(x)\,dx = 0 \quad\Rightarrow\quad 0 = \int_0^1\left(\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x + E\right)dx = \frac{1}{24} - \frac{1}{12} + \frac{1}{24} + E \quad\Rightarrow\quad E = 0$. So $B_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x$. $B_4(x) = \int B_3(x)\,dx = \int\left(\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x\right)dx = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 + F$. But $\int_0^1 B_4(x)\,dx = 0 \quad\Rightarrow\quad 0 = \int_0^1\left(\frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 + F\right)dx = \frac{1}{120} - \frac{1}{48} + \frac{1}{72} + F \quad\Rightarrow\quad F = -\frac{1}{720}$. So $B_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}$.

(b) By FTC2, $B_n(1) - B_n(0) = \int_0^1 B_n'(x)\,dx = \int_0^1 B_{n-1}(x)\,dx = 0$ for $n - 1 \geq 1$, by definition. Thus, $B_n(0) = B_n(1)$ for $n \geq 2$.

(c) We know that $B_n(x) = \dfrac{1}{n!}\displaystyle\sum_{k=0}^{n}\binom{n}{k}b_k x^{n-k}$. If we set $x = 1$ in this expression, and use the fact that $B_n(1) = B_n(0) = \dfrac{b_n}{n!}$ for $n \geq 2$, we get $b_n = \displaystyle\sum_{k=0}^{n}\binom{n}{k}b_k$. Now if we expand the right-hand side, we get

$b_n = \binom{n}{0}b_0 + \binom{n}{1}b_1 + \cdots + \binom{n}{n-2}b_{n-2} + \binom{n}{n-1}b_{n-1} + \binom{n}{n}b_n$. We cancel the b_n terms, move the b_{n-1} term to the LHS and divide by $-\binom{n}{n-1} = -n$: $b_{n-1} = -\frac{1}{n}\left[\binom{n}{0}b_0 + \binom{n}{1}b_1 + \cdots + \binom{n}{n-2}b_{n-2}\right]$ for $n \geq 2$, as required.

(d) We use mathematical induction. For $n = 0$: $B_0(1-x) = 1$ and $(-1)^0 B_0(x) = 1$, so the equation holds for $n = 0$ since $b_0 = 1$. Now if $B_k(1-x) = (-1)^k B_k(x)$, then since $\frac{d}{dx}B_{k+1}(1-x) = B'_{k+1}(1-x)\frac{d}{dx}(1-x) = -B_k(1-x)$, we have $\frac{d}{dx}B_{k+1}(1-x) = (-1)(-1)^k B_k(x) = (-1)^{k+1} B_k(x)$. Integrating, we get $B_{k+1}(1-x) = (-1)^{k+1} B_{k+1}(x) + C$. But the constant of integration must be 0, since if we substitute $x = 0$ in the equation, we get $B_{k+1}(1) = (-1)^{k+1} B_{k+1}(0) + C$, and if we substitute $x = 1$ we get $B_{k+1}(0) = (-1)^{k+1} B_{k+1}(1) + C$, and these two equations together imply that $B_{k+1}(0) = (-1)^{k+1}\left[(-1)^{k+1} B_{k+1}(0) + C\right] + C = B_{k+1}(0) + 2C \quad\Leftrightarrow\quad C = 0$. So the equation holds for all n, by induction. Now if the power of -1 is odd, then we have $B_{2n+1}(1-x) = -B_{2n+1}(x)$. In particular, $B_{2n+1}(1) = -B_{2n+1}(0)$. But from part (b), we know that $B_k(1) = B_k(0)$ for $k > 1$. The only possibility is that $B_{2n+1}(0) = B_{2n+1}(1) = 0$ for all $n > 0$, and this implies that $b_{2n+1} = (2n+1)!\,B_{2n+1}(0) = 0$ for $n > 0$.

(e) From part (a), we know that $b_0 = 0!\,B_0(0) = 1$, and similarly $b_1 = -\dfrac{1}{2}$, $b_2 = \dfrac{1}{6}$, $b_3 = 0$ and $b_4 = -\dfrac{1}{30}$.

We use the formula to find

$$b_6 = b_{7-1} = -\frac{1}{7}\left[\binom{7}{0}b_0 + \binom{7}{1}b_1 + \binom{7}{2}b_2 + \binom{7}{3}b_3 + \binom{7}{4}b_4 + \binom{7}{5}b_5\right]$$

The b_3 and b_5 terms are 0, so this is equal to

$$-\frac{1}{7}\left[1 + 7\left(-\frac{1}{2}\right) + \frac{7\cdot 6}{2\cdot 1}\left(\frac{1}{6}\right) + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)\right] = -\frac{1}{7}\left(1 - \frac{7}{2} + \frac{7}{2} - \frac{7}{6}\right) = \frac{1}{42}$$

Similarly,

$$\begin{aligned}
b_8 &= -\frac{1}{9}\left[\binom{9}{0}b_0 + \binom{9}{1}b_1 + \binom{9}{2}b_2 + \binom{9}{4}b_4 + \binom{9}{6}b_6\right] \\
&= -\frac{1}{9}\left[1 + 9\left(-\frac{1}{2}\right) + \frac{9\cdot 8}{2\cdot 1}\left(\frac{1}{6}\right) + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\left(-\frac{1}{30}\right) + \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\left(\frac{1}{42}\right)\right] \\
&= -\frac{1}{9}\left(1 - \frac{9}{2} + 6 - \frac{21}{5} + 2\right) = -\frac{1}{30}
\end{aligned}$$

Now we can calculate

$$\begin{aligned}
B_5(x) &= \frac{1}{5!}\sum_{k=0}^{5}\binom{5}{k}b_k x^{5-k} \\
&= \frac{1}{120}\left[x^5 + 5\left(-\frac{1}{2}\right)x^4 + \frac{5\cdot 4}{2\cdot 1}\left(\frac{1}{6}\right)x^3 + 5\left(-\frac{1}{30}\right)x\right] \\
&= \tfrac{1}{120}\left(x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x\right)
\end{aligned}$$

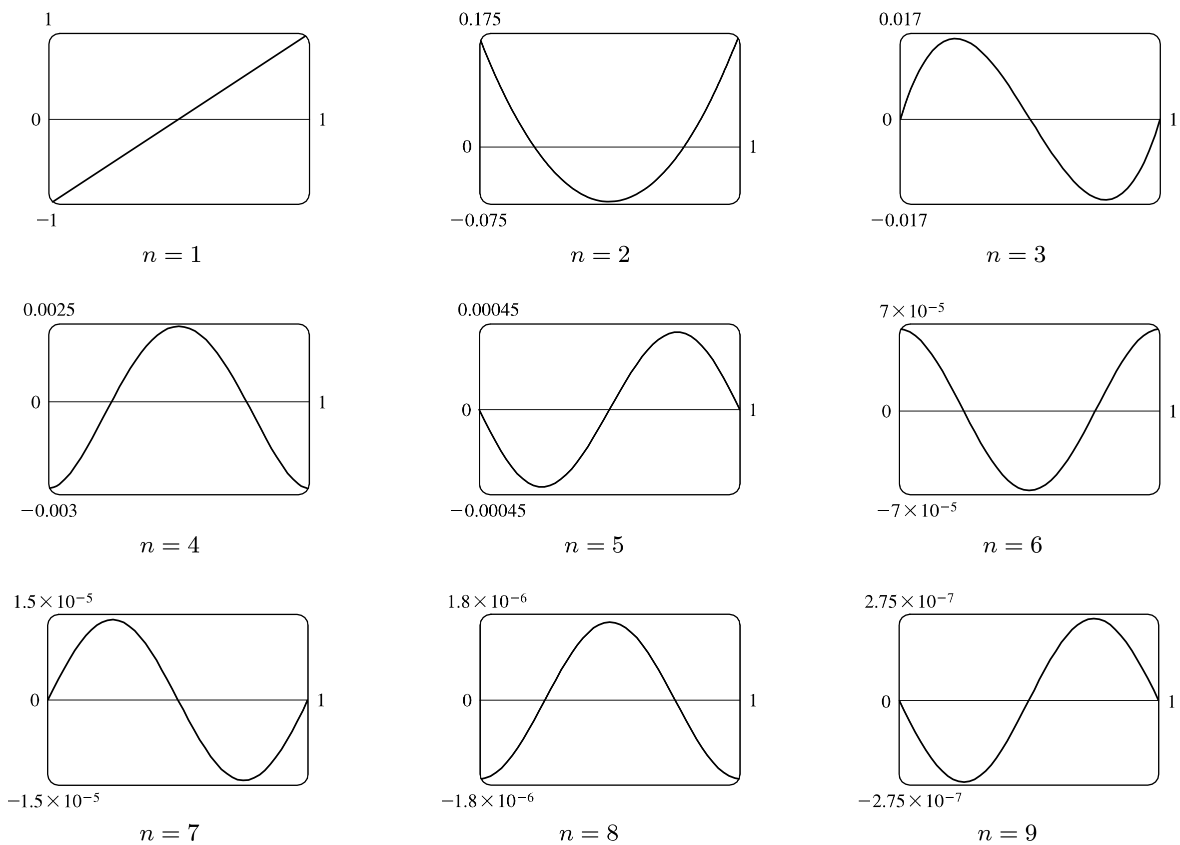
$$B_6(x) = \frac{1}{720}\left[x^6 + 6\left(-\frac{1}{2}\right)x^5 + \frac{6\cdot 5}{2\cdot 1}\left(\frac{1}{6}\right)x^4 + \frac{6\cdot 5}{2\cdot 1}\left(-\frac{1}{30}\right)x^2 + \frac{1}{42}\right]$$
$$= \tfrac{1}{720}\left(x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42}\right)$$

$$B_7(x) = \frac{1}{5040}\left[x^7 + 7\left(-\frac{1}{2}\right)x^6 + \frac{7\cdot 6}{2\cdot 1}\left(\frac{1}{6}\right)x^5 + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)x^3 + 7\left(\frac{1}{42}\right)x\right]$$
$$= \tfrac{1}{5040}\left(x^7 - \tfrac{7}{2}x^6 + \tfrac{7}{2}x^5 - \tfrac{7}{6}x^3 + \tfrac{1}{6}x\right)$$

$$B_8(x) = \frac{1}{40{,}320}\left[x^8 + 8\left(-\frac{1}{2}\right)x^7 + \frac{8\cdot 7}{2\cdot 1}\left(\frac{1}{6}\right)x^6 + \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)x^4 + \frac{8\cdot 7}{2\cdot 1}\left(\frac{1}{42}\right)x^2 + \left(-\frac{1}{30}\right)\right]$$
$$= \tfrac{1}{40{,}320}\left(x^8 - 4x^7 + \tfrac{14}{3}x^6 - \tfrac{7}{3}x^4 + \tfrac{2}{3}x^2 - \tfrac{1}{30}\right)$$

$$B_9(x) = \frac{1}{362{,}880}\left[x^9 + 9\left(-\frac{1}{2}\right)x^8 + \frac{9\cdot 8}{2\cdot 1}\left(\frac{1}{6}\right)x^7 + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)x^5 + \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\left(\frac{1}{42}\right)x^3 + 9\left(-\frac{1}{30}\right)x\right]$$
$$= \tfrac{1}{362{,}880}\left(x^9 - \tfrac{9}{2}x^8 + 6x^7 - \tfrac{21}{5}x^5 + 2x^3 - \tfrac{3}{10}x\right)$$

(f)



There are four basic shapes for the graphs of B_n (excluding B_1), and as n increases, they repeat in a cycle of four. For $n = 4m$, the shape resembles that of the graph of $-\cos 2\pi x$; For $n = 4m + 1$, that of $-\sin 2\pi x$; for $n = 4m + 2$, that of $\cos 2\pi x$; and for $n = 4m + 3$, that of $\sin 2\pi x$.

(g) For $k = 0$: $B_1(x+1) - B_1(x) = x + 1 - \frac{1}{2} - \left(x - \frac{1}{2}\right) = 1$, and $\dfrac{x^0}{0!} = 1$, so the equation holds for $k = 0$. We now assume that $B_n(x+1) - B_n(x) = \dfrac{x^{n-1}}{(n-1)!}$. We integrate this equation with respect to x:

$\displaystyle\int [B_n(x+1) - B_n(x)]\, dx = \int \frac{x^{n-1}}{(n-1)!}\, dx$. But we can evaluate the LHS using the definition $B_{n+1}(x) = \int B_n(x)\, dx$, and the RHS is a simple integral. The equation becomes

$B_{n+1}(x+1) - B_{n+1}(x) = \dfrac{1}{(n-1)!}\left(\dfrac{1}{n}x^n\right) = \dfrac{1}{n!}x^n$, since by part (b) $B_{n+1}(1) - B_{n+1}(0) = 0$, and so the constant of integration must vanish. So the equation holds for all k, by induction.

(h) The result from part (g) implies that $p^k = k!\,[B_{k+1}(p+1) - B_{k+1}(p)]$. If we sum both sides of this equation from $p = 0$ to $p = n$ (note that k is fixed in this process), we get $\sum\limits_{p=0}^{n} p^k = k! \sum\limits_{p=0}^{n} [B_{k+1}(p+1) - B_{k+1}(p)]$. But the RHS is just a telescoping sum, so the equation becomes $1^k + 2^k + 3^k + \cdots + n^k = k!\,[B_{k+1}(n+1) - B_{k+1}(0)]$. But from the definition of Bernoulli polynomials (and using the Fundamental Theorem of Calculus), the RHS is equal to $k! \int_0^{n+1} B_k(x)\, dx$.

(i) If we let $k = 3$ and then substitute from part (a), the formula in part (h) becomes

$$\begin{aligned}
1^3 + 2^3 + \cdots + n^3 &= 3!\,[B_4(n+1) - B_4(0)] \\
&= 6\left[\tfrac{1}{24}(n+1)^4 - \tfrac{1}{12}(n+1)^3 + \tfrac{1}{24}(n+1)^2 - \tfrac{1}{720} - \left(\tfrac{1}{24} - \tfrac{1}{12} + \tfrac{1}{24} - \tfrac{1}{720}\right)\right] \\
&= \frac{(n+1)^2[1 + (n+1)^2 - 2(n+1)]}{4} = \frac{(n+1)^2[1 - (n+1)]^2}{4} = \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

(j) $$\begin{aligned}
1^k + 2^k + 3^k + \cdots + n^k &= k! \int_0^{n+1} B_k(x)\, dx \qquad \text{[by part (h)]} \\
&= k! \int_0^{n+1} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} b_j x^{k-j}\, dx = \int_0^{n+1} \sum_{j=0}^{k} \binom{k}{j} b_j x^{k-j}\, dx
\end{aligned}$$

Now view $\sum\limits_{j=0}^{k} \dbinom{k}{j} b_j x^{k-j}$ as $(x+b)^k$, as explained in the problem. Then

$$1^k + 2^k + 3^k + \cdots + n^k \text{ “}=\text{” } \int_0^{n+1} (x+b)^k\, dx = \left[\frac{(x+b)^{k+1}}{k+1}\right]_0^{n+1} = \frac{(n+1+b)^{k+1} - b^{k+1}}{k+1}$$

(k) We expand the RHS of the formula in (j), turning the b^i into b_i, and remembering that $b_{2i+1} = 0$ for $i > 0$:

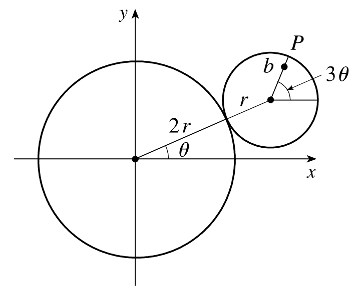
$$\begin{aligned}
1^5 + 2^5 + \cdots + n^5 &= \tfrac{1}{6}\left[(n+1+b)^6 - b^6\right] \\
&= \tfrac{1}{6}\left[(n+1)^6 + 6(n+1)^5 b_1 + \tfrac{6\cdot 5}{2\cdot 1}(n+1)^4 b_2 + \tfrac{6\cdot 5}{2\cdot 1}(n+1)^2 b_4\right] \\
&= \tfrac{1}{6}\left[(n+1)^6 - 3(n+1)^5 + \tfrac{5}{2}(n+1)^4 - \tfrac{1}{2}(n+1)^2\right] \\
&= \tfrac{1}{12}(n+1)^2\left[2(n+1)^4 - 6(n+1)^3 + 5(n+1)^2 - 1\right] \\
&= \tfrac{1}{12}(n+1)^2\,[(n+1) - 1]^2\left[2(n+1)^2 - 2(n+1) - 1\right] = \tfrac{1}{12}n(n+1)^2(2n^2 + 2n - 1)
\end{aligned}$$

E Exercises Chapter 6

1. The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi\,[f(x)]^2\,dx$. Hence, we are given that $b^2 = \int_0^b \pi\,[f(x)]^2\,dx$ for all $b > 0$. Differentiating both sides of this equation using the Fundamental Theorem of Calculus gives $2b = \pi\,[f(b)]^2 \;\Rightarrow\; f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

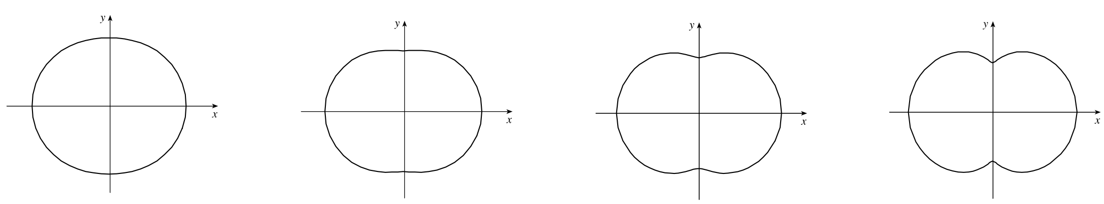
E Exercises Chapter 10

1. (a) Since the smaller circle rolls without slipping around C, the amount of arc traversed on C ($2r\theta$ in the figure) must equal the amount of arc of the smaller circle that has been in contact with C. Since the smaller circle has radius r, it must have turned through an angle of $2r\theta/r = 2\theta$. In addition to turning through an angle 2θ, the little circle has rolled through an angle θ against C. Thus, P has turned through an angle of 3θ as shown in the figure. (If the little circle had turned through an angle of 2θ with its center pinned to the x-axis, then P would have turned only 2θ instead of 3θ.



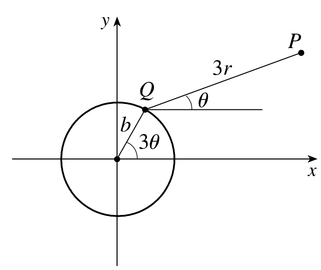
The movement of the little circle around C adds θ to the angle.) From the figure, we see that the center of the small circle has coordinates $(3r\cos\theta, 3r\sin\theta)$. Thus, P has coordinates (x, y), where $x = 3r\cos\theta + b\cos 3\theta$ and $y = 3r\sin\theta + b\sin 3\theta$.

(b)

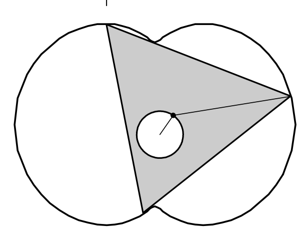


$b = \frac{1}{5}r$ $\quad b = \frac{2}{5}r$ $\quad b = \frac{3}{5}r$ $\quad b = \frac{4}{5}r$

(c) The diagram gives an alternate description of point P on the epitrochoid. Q moves around a circle of radius b, and P rotates one-third as fast with respect to Q at a distance of $3r$. Place an equilateral triangle with sides of length $3\sqrt{3}\,r$ so that its centroid is at Q and one vertex is at P. (The distance from the centroid to a vertex is $\frac{1}{\sqrt{3}}$ times the length of a side of the equilateral triangle.)



As θ increases by $\frac{2\pi}{3}$, the point Q travels once around the circle of radius b, returning to its original position. At the same time, P (and the rest of the triangle) rotate through an angle of $\frac{2\pi}{3}$ about Q, so P's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by P is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.



(d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is $3r$, so it has radius $6r$. To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point P, there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when P is on the y-axis, so as long as the diameter of the rotor (which is $3\sqrt{3}\,r$) is less than the distance between the y-intercepts, the rotor will fit. The y-intercepts occur when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2} \Rightarrow$ $y = \pm(3r - b)$, so the distance between the intercepts is $6r - 2b$, and the rotor will fit if $3\sqrt{3}r \le 6r - 2b \Leftrightarrow$ $b \le \frac{3(2-\sqrt{3})}{2}r$.

E Exercises Chapter 11

1. (a) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\left(2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2} = 2\left(2\left(2\sin\frac{\theta}{8}\cos\frac{\theta}{8}\right)\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2}$

$$= \cdots = 2\left(2\left(2\left(\cdots\left(2\left(2\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}\right)\cos\frac{\theta}{2^{n-1}}\right)\cdots\right)\cos\frac{\theta}{8}\right)\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2}$$

$$= 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}$$

(b) $\sin\theta = 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n} \quad\Leftrightarrow\quad \frac{\sin\theta}{\theta}\cdot\frac{\theta/2^n}{\sin(\theta/2^n)} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}$.

Now we let $n \to \infty$, using $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ with $x = \frac{\theta}{2^n}$:

$$\lim_{n\to\infty}\left[\frac{\sin\theta}{\theta}\cdot\frac{\theta/2^n}{\sin(\theta/2^n)}\right] = \lim_{n\to\infty}\left[\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}\right] \quad\Leftrightarrow\quad \frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots$$

(c) If we take $\theta = \frac{\pi}{2}$ in the result from part (b) and use the half-angle formula $\cos x = \sqrt{\frac{1}{2}(1+\cos 2x)}$ (see Formula 17a in Appendix D), we get

$$\frac{\sin\pi/2}{\pi/2} = \cos\tfrac{\pi}{4}\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}\sqrt{\frac{\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}+1}{2}}\sqrt{\frac{\sqrt{\frac{\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}+1}{2}}+1}{2}}\cdots \quad\Rightarrow$$

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}+1}{2}}\cdots = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\sqrt{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}+1}{2}}\cdots$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$